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Relativistic Quantum Mechanics over Stochastic Phase Space[1]

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Stochastic quantum mechanics is a quantum theory in which the basic limitations of real-world measuring instruments, due to their intrinsically quantum nature, are taken into account. Among other things this leads to a new operational definition of space-time, called quantum space-time. Fundamental to this approach is the formulation of quantum mechanics over phase space rather than just over position or momentum space. A concept of extended particle is a natural outgrowth of this development. Gauge and internal symmetry have a natural place within the theory, and preliminary computations combining some old ideas due to Born with more recent ideas on symmetry breaking suggest that the theory could lead to a mass formula compatible with known data on the low-lying baryons.

1. INTRODUCTION

Classical physics is said to be deterministic because the equations which govern the evolution of a classical physical system typically uniquely determine the state of the system at all future times, once the state of the system has been precisely specified at some initial time. As long as one is willing to permit idealized measuring instruments (measuring rods for example, on which perfectly exact and sharp markings are engraved), then the determination of the future state of a system from an initially precise state is not a problem. However, any initial imprecision in the specification of the state of a classical system will result in (generally greater) future imprecision, and as is well known, perfectly precise measuring instruments do not exist in reality. So the result of a realistic measurement of a classical observable should in fact be a value of the observable together with a

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probability distribution, characteristic of the measuring device used, specifying what readings one ought to expect.

Quantum physics, in its conventional formulation, allows that certain pairs of observables may not be simultaneously prescribed or measured with arbitrary accuracy in accordance with Heisenberg's uncertainty principle. But as far as a single observable is concerned, the usual approach permits arbitrarily precise specification of its value. One flaw with this approach is that real instruments are constructed of quantum mechanical and not classical mechanical point particles, and therefore in the limit of the ultimate instrument, namely, a quantum mechanical particle itself, one should not expect to recover arbitrarily high precision. Despite this drawback, however, the imprecisions of a quantum instrument (for example, a particle used as a position marker) are not arbitrary but must be consistent with the uncertainty principle (so that a quantum particle used to measure position and velocity would possess intrinsic but well-defined limitations). The description of quantum mechanics which takes into account these fundamental difficulties has been achieved and has become known as stochastic quantum mechanics (for a review of the entire program see Prugovečki, 1984).

In this paper we give a brief review of the basic notions of the theory of stochastic quantum mechanics together with an account of some recent developments. Section 2 concerns the covariance properties of the stochastic phase space concept which underlies the entire approach. The reciprocity principle of Born and its role within the theory is outlined in Section 3. The introduction of gauge freedom via the canonical commutation relations is described in Section 4, and in Section 5 we give some preliminary calculations leading to a mass formula which agrees well with experiment in the case of low-lying baryons.

2. STOCHASTIC PHASE SPACE

Suppose a simultaneous measurement of position $\mathbf{Q}$ and momentum $\mathbf{P}$ of a particle is to be carried out. The outcome of such a measurement will not be simply the coordinates $\mathbf{q}$ of position and $\mathbf{p}$ of momentum, but rather $\mathbf{q}$ and $\mathbf{p}$ together with confidence functions $\chi_{\mathbf{q}}(\mathbf{x})$ and $\hat{\chi}_{\mathbf{p}}(\mathbf{k})$ representing probability densities that $\mathbf{Q}, \mathbf{P}$ will have values $\mathbf{x}, \mathbf{k}$ respectively such that the uncertainties of $\mathbf{Q}$ and $\mathbf{P}$ (equalling the standard deviations of $\chi_{\mathbf{q}}$ and $\hat{\chi}_{\mathbf{p}}$, respectively) obey the uncertainty principle. In the optimal case, the "spreads" of $\chi_{\mathbf{q}}$ and $\hat{\chi}_{\mathbf{p}}$ will be in inverse proportion. In the limit of perfectly sharp position measurements the confidence function $\chi_{\mathbf{q}}(\mathbf{x})$ goes over into a delta function $\delta(\mathbf{x}-\mathbf{q})$ located at $\mathbf{q}$, whereas $\hat{\chi}_{\mathbf{p}}(\mathbf{k})$ becomes a constant density. And in the sharp momentum limit $\hat{\chi}_{\mathbf{p}}(\mathbf{k})$ becomes $\delta(\mathbf{k}-\mathbf{p})$ and $\chi_{\mathbf{q}}(\mathbf{x})$ a uniform distribution.

Since the uncertainty relations (which serve to determine the limits of accuracy attainable by quantum instruments) hold between conjugate observables, it is natural to formulate quantum mechanics over the space of such pairs of conjugate variables, namely, phase space. The nonrelativistic situation (Prugovečki, 1976; Ali and Prugovečki, 1977, 1983a) is as follows.

Consider a single spinless particle with fixed-time phase space

$$\Gamma = \{(\mathbf{q},\mathbf{p})\} = \mathbb{R}^6 \tag{1}$$

and Hilbert space over this phase space

$$L^2(\Gamma) = \left\{\psi : \int_\Gamma |\psi(\mathbf{q},\mathbf{p})|^2 d\mathbf{q}\, d\mathbf{p} < \infty\right\} \tag{2}$$

A realization of the canonical commutation relations

$$[Q^i, P^j] = i\hbar\delta^{ij}, \qquad [Q^i, Q^j] = 0 = [P^i, P^j] \tag{3}$$

is given by

$$Q^i\psi = q^i\psi + i\hbar\frac{\partial\psi}{\partial p^i}, \qquad P^i\psi = -i\hbar\frac{\partial\psi}{\partial q^i} \tag{4}$$

The wave functions may be made time dependent according to the free evolution

$$\psi(\mathbf{q},\mathbf{p},t) = \left[\exp\left(-\frac{i}{\hbar}H_0 t\right)\psi\right](\mathbf{q},\mathbf{p}) \tag{5}$$

where $H_0 = (1/2m)\mathbf{P}^2$ is the free Hamiltonian. Thus we have a unitary ray representation of the Galilei group given by

$$\begin{aligned}[U(b,\mathbf{a},\mathbf{v},R)\psi](\mathbf{q},\mathbf{p},t) &= \exp\left\{\frac{i}{\hbar}\left[m\mathbf{v}\cdot(\mathbf{q}-\mathbf{a}) - \frac{1}{2}m\mathbf{v}^2(t-b)\right]\right\} \\ &\quad \times \psi\left(R^{-1}\{\mathbf{q}-\mathbf{a}-\mathbf{v}(t-b)\}, R^{-1}(\mathbf{p}-m\mathbf{v}), t-b\right)\end{aligned} \tag{6}$$

where $(b,\mathbf{a},\mathbf{v},R)$ represents an element of the Galilei group corresponding to time translation by b, space translation by $\mathbf{a}$, a boost by $\mathbf{v}$, and a rotation in space by R. Note that

$$U(0,\mathbf{a},\mathbf{0},I)=\exp\left(-\frac{i}{\hbar}\mathbf{a}\cdot\mathbf{P}\right) \tag{7a}$$

$$U(0,\mathbf{0},\mathbf{v},I)=\exp\left(\frac{i}{\hbar}m\mathbf{v}\cdot\mathbf{Q}\right) \tag{7b}$$

so that $\mathbf{P}$ and $\mathbf{Q}$, respectively, generate space translations and boosts (or momentum translations).

Although the representation (6) is reducible, it can be related to the usual configuration representation as follows. Given a rotationally invariant $\xi(\mathbf{x})\in L^2(\mathbb{R}^3)$ with $\|\xi\|^2=(2\pi\hbar)^{-3}$, let

$$\xi_{\mathbf{q},\mathbf{p}}(\mathbf{x})=\left[\underset{\substack{\text{config.}\\ \text{space}}}{U}\left(0,\mathbf{q},\frac{1}{m}\mathbf{p},I\right)\xi\right](\mathbf{x})=\exp\left\{\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}-\mathbf{q})\right\}\xi(\mathbf{x}-\mathbf{q}) \tag{8}$$

where $U_{\text{config. space}}$ means U restricted to functions of the first variable only. Now define

$$W_\xi:\psi(\mathbf{x})\mapsto\psi(\mathbf{q},\mathbf{p})=\int_{\mathbb{R}^3}\xi^*_{\mathbf{q},\mathbf{p}}(\mathbf{x})\psi(\mathbf{x})\,d\mathbf{x} \tag{9}$$

which maps $L^2(\mathbb{R}^3)=\{\text{sharp configuration space wave functions}\}$ isometrically into $L^2(\Gamma)=\{\text{phase space wave functions}\}$ and intertwines the representations U on $L^2(\Gamma)$ and $U_{\text{config. space}}$ on $L^2(\mathbb{R}^3)$. Thus U restricted to $W_\xi L^2(\mathbb{R}^3)\subset L^2(\Gamma)$ is unitarily equivalent to $U_{\text{config. space}}$. The correspondence $\psi(\mathbf{x})$ to $\psi(\mathbf{q},\mathbf{p})$ via W_ξ also has the so-called marginality properties:

$$\int_{\mathbb{R}^3}|\psi(\mathbf{q},\mathbf{p})|^2\,d\mathbf{p}=\int_{\mathbb{R}^3}\chi^\xi_{\mathbf{q}}(\mathbf{x})|\psi(\mathbf{x})|^2\,d\mathbf{x} \tag{10a}$$

$$\int_{\mathbb{R}^3}|\psi(\mathbf{q},\mathbf{p})|^2\,d\mathbf{q}=\int_{\mathbb{R}^3}\hat{\chi}^\xi_{\mathbf{p}}(\mathbf{k})|\tilde{\psi}(\mathbf{k})|^2\,d\mathbf{k} \tag{10b}$$

where

$$\chi^\xi_{\mathbf{q}}(\mathbf{x})=(2\pi\hbar)^3|\xi(\mathbf{x}-\mathbf{q})|^2$$

$$\hat{\chi}^\xi_{\mathbf{p}}(\mathbf{k})=(2\pi\hbar)^3|\tilde{\xi}(\mathbf{k}-\mathbf{p})|^2$$

and $\sim$ denotes Fourier transform.

The physical interpretation is this. If $|\psi(\mathbf{x})|^2$ is the probability density that a point particle be detected at $\mathbf{x}$ using a perfectly accurate instrument, and if a realistic apparatus produces a density $\chi_{\mathbf{q}}^{\xi}(\mathbf{x}) = \chi_{\mathbf{0}}^{\xi}(\mathbf{x}-\mathbf{q})$ for reading $\mathbf{q}$ when a particle is present at $\mathbf{x}$, then (10a) gives the probability of a response $\mathbf{q}$ when a point particle in a state ψ is present. Therefore $|\psi(\mathbf{q},\mathbf{p})|^2$ becomes a probability density on stochastic phase space

$$\Gamma_{\xi} = \left\{\left((\mathbf{q}, \chi_{\mathbf{q}}^{\xi}), (\mathbf{p}, \hat{\chi}_{\mathbf{p}}^{\xi})\right)\right\}$$

The relativistic construction is carried out analogously by replacing the phase space representation of the Galilei group with that of the Poincaré group (see Prugovečki, 1978a, b; Ali and Prugovečki, 1983b).

Instead of the canonical commutation relations (3) we consider the relativistic canonical commutation relations (RCCR's):

$$[Q^{\mu}, P^{\nu}] = -i\hbar g^{\mu\nu}, \qquad [Q^{\mu}, Q^{\nu}] = 0 = [P^{\mu}, P^{\nu}] \tag{11}$$

where $(g^{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ with Greek indices ranging through $0,1,2,3$. In analogy to $\tilde{\xi}(\mathbf{k})$ we now have $\tilde{\eta}(k) \in L^2(V_m^{\pm}, d\Omega_m)$, the functions on the future $(+)$ or past $(-)$ mass-m shell $V_m^{\pm} = \{k\colon k^{\mu}k_{\mu} = m^2c^2\}$ square integrable with respect to the measure $d\Omega_m(k) = \delta(k^2 - m^2c^2)d^4k = d\mathbf{k}/k^0$. The function $\tilde{\eta}$ is normalized to $\|\tilde{\eta}\|^2 = 2mc(2\pi\hbar)^{-3}$, and again by analogy with $\tilde{\xi}_{\mathbf{q},\mathbf{p}}$ one has $\tilde{\eta}_{q,p}$ which is $\tilde{\eta}$ boosted to 4-velocity $(1/m)p$ and translated in space time by q. One has a mapping

$$W_{\eta}\colon L^2(V_m^{\pm}, d\Omega_m) \to L^2(\Sigma_m^{\pm}, d\Sigma_m^{\pm})$$

$$\tilde{\phi}(k) \mapsto \tilde{\phi}(q,p) = \int_{V_m^{\pm}} \tilde{\eta}_{q,p}^{*}(k)\tilde{\phi}(k)\, d\Omega_m(k) \tag{12}$$

where $\Sigma_m^{\pm} = \sigma \times V_m^{\pm}$

$$\sigma = \text{spacelike hyperplane of space time}$$

and $d\Sigma_m = 2\,\mathrm{sgn}(p^0)p^{\mu}\, d\sigma_{\mu}(q)\, d\Omega_m(p)$, the measure on $\Sigma_m^{\pm}$ which reduces in the special case of $\sigma = \{q^0 = \text{const.}\}$ to $d\mathbf{q}\, d\mathbf{p}$. The image $L^2(\Sigma_{m,\eta}^{\pm})$ of $L^2(V_m^{\pm})$ by W_{η} is an irreducible subspace of $L^2(\Sigma_m^{\pm})$ supporting a Poincaré representation unitarily equivalent to the usual momentum space representation.

One also has a conserved $(\mathrm{div} j_{\eta} = 0)$, covariant and bonafide probability current

$$j_{\eta}^{\mu}(q) = \int_{V_m^{\pm}} \frac{2\,\mathrm{sgn}(p^0)p^{\mu}}{m} |\tilde{\phi}(q,p)|^2\, d\Omega_m(p) \tag{13}$$

when $\tilde{\eta}$ is real valued.

Finally, it should be noted that all of the physics resides in the relativistic free propagator

$$K_\eta(q', p';\ q, p) = \langle \tilde{\eta}_{q',p'} | \tilde{\eta}_{q,p} \rangle_{V_m^\pm} \tag{14}$$

because

$$K_\eta^*(q, p;\ q', p') = K_\eta(q', p', q, p) \tag{15a}$$

$$K_\eta(q', p';\ q, p) = \int_{\Sigma_m^\pm} K_\eta(q', p';\ q'', p'') K_\eta(q'', p'';\ q, p)\, d\Sigma_m(q'', p'') \tag{15b}$$

and because in the framework of stochastic quantum mechanics, $(2\pi\hbar)^3 \times |K_\eta(q'', p'';\ q', p')|^2$ represents the probability that an extended quantum mechanical particle (characterized by η) located at (q', p') will reach (q'', p'').

3. RECIPROCITY PRINCIPLE AND SUBSEQUENT DEVELOPMENTS

Motivated by symmetry arguments in quantum theory, Born (1938) was led to consider the possibility that the usual Minkowski metric on space-time should be generalized to permit a velocity dependence. In fact he suggested that the line element

$$ds^2 = dq^\mu dq_\mu + dp^\mu dp_\mu \tag{16}$$

which is symmetrical in q and p would be a more appropriate one over small distances where quantum effects are dominant.

Shortly thereafter, Born (1939) and Landé (1939), in seeking to avoid the divergences of quantum field theory associated with point particles, postulated a symmetric counterpart to the Klein–Gordon equation,

$$\left(-\hbar^2 \frac{\partial^2}{\partial q_\mu \partial q^\mu} - m^2c^2\right)\psi = 0 \tag{17}$$

namely, the Born–Landé equation,

$$\left(-\hbar^2 \frac{\partial}{\partial p_\mu \partial p^\mu} + l^2\right)\phi = 0 \tag{18}$$

which is obtainable from the Klein–Gordon equation essentially by replacing q by p and mc by l. This l should then represent a fundamental length of the field thereby giving rise to a notion of particle extension.

The reciprocity idea (or q vs. p symmetry) was ultimately refined by Born (1949) as follows. First, one should normalize the position and momentum variables

$$q \mapsto q/l_0, \qquad p \mapsto p/m_0 c \tag{19}$$

where l_0 and m_0 represent a fundamental length and mass, respectively. The reciprocity principle would then require that the fundamental laws of physics be invariant under the reciprocity transformation ρ defined on normalized variables q, p by

$$\begin{aligned} q &\mapsto p \\ p &\mapsto -q. \end{aligned} \tag{20}$$

One justification for considering this transformation aside from aesthetic symmetry grounds is that it leaves Hamilton's equations invariant (ρ is canonical); that is,

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{21}$$

are reciprocally invariant, or more geometrically that the symplectic structure

$$\Omega = dp \wedge dq \tag{22}$$

is ρ invariant (i.e., $dp \wedge dq \mapsto (-dq) \wedge dp = dp \wedge dq$). Moreover, the canonical commutation relations (relativistic or nonrelativistic) are reciprocally invariant

$$[q, p] = i\hbar \mapsto [p, -q] = ih \Leftrightarrow [q, p] = i\hbar \tag{23}$$

as also are the expressions (relativistic and nonrelativistic) for angular momentum. One interpretation of the reciprocity transformation is that it selects from the space of generalized phase space variables a set of "positions" p and "momenta" p which are canonically conjugate and in involution (the "positions" Poisson-commuting with one another and likewise for the "momenta"). Such variables, either the q's or the p's, naturally correspond in quantum mechanics to complete sets of commuting observables.

So the reciprocity principle can be regarded as the prequantum specification of "physical" positions and momenta from among the generalized position and momentum variables of phase space.

In place of the Klein–Gordon operator $P^\mu P_\mu$ Born suggested the use of the so-called quantum metric operator

$$D^2 = Q^\mu Q_\mu + P^\mu P_\mu \tag{24}$$

reciprocally symmetric with respect to Q^μ and P^μ, where Q^μ, P^μ satisfy the normalized RCCR's (11)

$$[Q^\mu, P^\nu] = -i\omega_0 g^{\mu\nu}, \qquad [Q^\mu, Q^\nu] = 0 = [P^\mu, P^\nu] \tag{25}$$

with $\omega_0 = \hbar / l_0 m_0 c$ a dimensionless Planck's constant. The idea was to identify elementary particles with eigenfunctions of D^2 whose eigenvalues would somehow embody both the masses and radii of the allowed solutions. Besides the symmetrical form of D^2 as suggested by (16), one may argue that the Q^2 term of D^2 is dominant on the cosmological scale where in relative terms energies are much smaller than lengths. Moreover, the P^2 term would be expected to prevail on the microscopic scale characteristic of high energy, and so there was hope that the quantum metric operator, embodying both the macro- and the microscopic might lead to a reconciliation of quantum theory and relativity.

These developments led directly to the nonlocal models of Yukawa (1950a, b; 1953) in which for two particles one would define

$$X^\mu = \tfrac{1}{2}(x_1^\mu + x_2^\mu), \qquad r^\mu = x_1^\mu - x_2^\mu \tag{26}$$

as external and internal positions, respectively, and then postulate a free field equation of the form

$$\left\{ -\hbar^2 \frac{\partial^2}{\partial X_\mu \partial X^\mu} + F\left(r^\mu r_\mu, \frac{\partial^2}{\partial r_\mu \partial r^\mu}, r^\mu \frac{\partial}{\partial r^\mu} \right) \right\} \phi(X, r) = 0 \tag{27}$$

Thus began the study of bi- and multilocal models which led in particular (when $F = r^\mu r_\mu$) to the relativistic harmonic oscillator models (see Takabayasi, 1979, for a review).

Within the stochastic quantum theory, Prugovečki (1981a, b) suggested the following stochastic analog of Born's eigenvalue equation:

$$D^2 K_{B,A} = \lambda_{B,A} K_{B,A} \tag{28}$$

where D^2 is as in (24), (25) and $K_{B,A}$ is the propagator (14) or exciton transition amplitude between a ground state A and an excited state B. From the form of $K_{B,A}$, namely,

$$K_{B,A}(q', p'; q, p)$$
$$= \int \exp\left\{\frac{i}{\omega_A}\left(q' - \frac{m_B}{m_A}q\right)\cdot k\right\}\hat{\eta}_A(m_A c(p' - k))\eta_B(u)\delta(k^2 - 1)\,d^4k \tag{29}$$

where $\eta_B(u) = \hat{\eta}_B(m_A cu)$, $u = p - (m_B/m_A)k$, it follows that (Brooke and Prugovečki, 1983)

$$\lambda_{B,A} = -2\omega_A(2 + n_0 + 2n + J_B) \tag{30}$$

in the case of integral spin, and for even(+) and odd(−) parity

$$\lambda^{\pm}_{B,A} = -2\omega_A(2 + n_0 + 2n + J_B \mp \tfrac{1}{2}) \tag{31}$$

in the case of half-integral spin. Here $\omega_A = \hbar/l_A m_A c$, J_B is the spin of the excited state B and n_0, $n = 0, 1, 2, \ldots$ are quantum numbers parametrizing the excited states.

Interestingly, the connection between (27) and (28) is very close since equation (28) for the exciton transition amplitude eigenvalue equation is equivalent to

$$\left\{-\omega_A^2 \frac{\partial^2}{\partial u_\mu \partial u^\mu} + u^\mu u_\mu\right\}\eta_B(u) = \lambda_{B,A}\eta_B(u) \tag{32}$$

with u as in (29). For further discussion of this point see Brooke and Guz (1983a).

4. RELATIVISTIC CANONICAL COMMUTATION RELATIONS AND GAUGE FREEDOM

The relativistic canonical commutation relations (11), which in the relativistic situation lie at the heart of quantum mechanics, can be employed to introduce gauge freedom into the stochastic quantum theory. The RCCR's of course contain a "time-energy" commutation relation requiring the existence of a time operator. For a discussion of such issues within the present context see Prugovečki (1982).

Use of the RCCR's to introduce gauge freedom was suggested by Caianiello (1980a, b) in a general program to unite geometry and quantum mechanics. Realizations of the RCCR's as first-order differential operators over phase space variables contained terms interpretable as connection coefficients arising from a metric of signature $(1,-1,-1,-1,1,-1,-1,-1)$ on phase space, so that quantum mechanics could be described as a kind of curvature of phase space.

The development described here can be found in Brooke and Prugovečki (1982) and Brooke and Guz (1983b).

We seek solutions of the RCCR's

$$[Q_\mu, P_\nu] = -i\omega_0 g_{\mu\nu}, \qquad [Q_\mu, Q_\nu] = 0 = [P_\mu, P_\nu] \tag{33}$$

where $\omega_0 = \hbar/l_0 m_0 c$, in the following form:

$$Q_\mu = -i\omega_0\left\{\frac{\partial}{\partial p^\mu} + \Phi_\mu(q,p)\right\} \tag{34}$$

$$P_\mu = i\omega_0\left\{\frac{\partial}{\partial q^\mu} + \Psi_\mu(q,p)\right\} \tag{34}$$

For scalar-valued functions Φ_μ, Ψ_μ one finds (33) to be equivalent to

$$\frac{\partial \Psi_\nu}{\partial p^\mu} - \frac{\partial \Phi_\mu}{\partial q^\nu} = -\frac{i}{\omega_0} g_{\mu\nu} \tag{35a}$$

$$\frac{\partial \Phi_\nu}{\partial p^\mu} - \frac{\partial \Phi_\mu}{\partial p^\nu} = 0 \tag{35b}$$

$$\frac{\partial \Psi_\nu}{\partial q^\mu} - \frac{\partial \Psi_\mu}{\partial q^\nu} = 0 \tag{35c}$$

The construction of the general solution proceeds by a simple differential geometric argument. Let λ denote the following differential 1-form (covariant vector field) on the space of the q's and p's:

$$\lambda = \Phi_\mu\, dp^\mu + \Psi_\mu\, dq^\mu \tag{36}$$

Taking the exterior derivative (curl), one has

$$d\lambda = \frac{1}{2}\left(\frac{\partial\Phi_\nu}{\partial p^\mu} - \frac{\partial\Phi_\mu}{\partial p^\nu}\right) dp^\mu \wedge dp^\nu + \left(\frac{\partial\Psi_\nu}{\partial p^\mu} - \frac{\partial\Phi_\mu}{\partial q^\nu}\right) dp^\mu \wedge dq^\nu$$
$$+ \frac{1}{2}\left(\frac{\partial\Psi_\nu}{\partial q^\mu} - \frac{\partial\Psi_\mu}{\partial q^\nu}\right) dq^\mu \wedge dq^\nu \tag{37}$$

with the result that the RCCR's (33) are equivalent to

$$d\lambda = -\frac{i}{\omega_0} g_{\mu\nu}\, dp^\mu \wedge dq^\nu = -\frac{i}{\omega_0}\, dp_\mu \wedge dq^\mu \tag{38}$$

with raising and lowering of indices via $g^{\mu\nu}, g_{\mu\nu}$. Now (38) may be written as

$$d\left(\lambda - \frac{i}{\omega_0} g_{\mu\nu} q^\mu\, dp^\nu\right) = 0 \tag{39}$$

which by the Poincaré lemma ($\operatorname{curl} A^\mu = 0 \Rightarrow A^\mu =$ gradient) is locally equivalent to

$$\lambda = \frac{i}{\omega_0} g_{\mu\nu} q^\mu\, dp^\nu + \frac{i}{\omega_0}\, d\omega \tag{40}$$

for some function ω. Thus

$$\Phi_\mu = \frac{i}{\omega_0}\left(g_{\mu\nu} q^\nu + \frac{\partial\omega}{\partial p^\mu}\right), \qquad \Psi_\mu = \frac{i}{\omega_0}\frac{\partial\omega}{\partial q^\mu} \tag{41}$$

and

$$Q_\mu = -i\omega_0 \frac{\partial}{\partial p^\mu} + q_\mu + \frac{\partial\omega}{\partial p^\mu} \tag{42a}$$

$$P_\mu = i\omega_0 \frac{\partial}{\partial q^\mu} - \frac{\partial\omega}{\partial q^\mu} \tag{42b}$$

Note that the realization (4) of the nonrelativistic CCR's is obtainable from (42) by setting $\omega = 0$, and moreover the similarity between the standard minimal coupling substitution $P_\mu \to P_\mu - A_\mu$ and (42b) with $A_\mu = \partial\omega/\partial q^\mu$. Of course from $A_\mu = \partial\omega/\partial q^\mu$ one has $F_{\mu\nu} = \partial A_\nu/\partial q^\mu - \partial A_\mu/\partial q^\nu = 0$. We see therefore that gauging the RCCR's by scalar-valued functions amounts to the usual wave function phase ambiguity associated with the introduction of electromagnetism in quantum mechanics.

Before generalizing this construction to general gauges, we shall consider RCCR's for "internal" position and momentum. In stochastic quantum mechanics one regards Q^μ, P^μ as fluctuating about mean values q^μ, p^μ which are observed in a measurement process according to the general principles of the theory. Therefore, one ought to consider as internal variables (with mean zero)

$$\hat{Q}^\mu = Q^\mu - q^\mu, \qquad \hat{P}^\mu = P^\mu - p^\mu \tag{43}$$

Using the realization (42) just found for Q^μ, P^μ one may easily verify that $\hat{Q}^\mu$, $\hat{P}^\mu$ satisfy the "internal" RCCR's,

$$[\hat{Q}^\mu, \hat{P}^\nu] = i\omega_0 g^{\mu\nu}, \qquad [\hat{Q}^\mu, \hat{Q}^\nu] = 0 = [\hat{P}^\mu, \hat{P}^\nu] \tag{44}$$

the only difference being a change of sign in the $\hat{Q}^\mu$, $\hat{P}^\mu$ relation.

Hence we now seek solutions of the internal RCCR's (44) in the case of general gauges. This means we look for solutions of the form

$$\hat{Q}_\mu = -i\omega_0\left\{\frac{\partial}{\partial p^\mu} + \Phi_\mu(q, p)\right\} \tag{45a}$$

$$\hat{P}_\mu = i\omega_0\left\{\frac{\partial}{\partial q^\mu} + \Psi_\mu(q, p)\right\} \tag{45b}$$

where Φ_μ, Ψ_μ take values in the Lie algebra of the gauge group $U(1)\times G^{\mathrm{INT}}$ [previously $U(1)$ with Lie algebra all real multiples of i].

According to the general Yang–Mills prescription, one constructs the Lie algebra valued 1-form [as in (36)]

$$\lambda = \Phi_\mu \, dp^\mu + \Psi_\mu \, dq^\mu \tag{46}$$

but to cope with non-Abelian G^{INT} one computes the exterior covariant derivative $\mathrm{D}\lambda$ by

$$\mathrm{D}\lambda = d\lambda + \tfrac{1}{2}[\lambda, \lambda] \tag{47}$$

the second nonlinear term in λ involving Lie bracket and exterior product.

Now the RCCR's (44) are equivalent to

$$\frac{\partial \Psi_\nu}{\partial p^\mu} - \frac{\partial \Phi_\mu}{\partial q^\nu} + [\Phi_\mu, \Psi_\nu] = \frac{i}{\omega_0} g_{\mu\nu} \tag{48a}$$

$$\frac{\partial \Phi_\nu}{\partial p^\mu} - \frac{\partial \Phi_\mu}{\partial p^\nu} + [\Phi_\mu, \Phi_\nu] = 0 \tag{48b}$$

$$\frac{\partial \Psi_\nu}{\partial q^\mu} - \frac{\partial \Psi_\mu}{\partial q^\nu} + [\Psi_\mu, \Psi_\nu] = 0 \tag{48c}$$

or because the components of $\mathsf{D}\lambda$ are precisely those on the left of (48), the RCCR's (44) are equivalent to

$$\mathsf{D}\lambda = \frac{i}{\omega_0} g_{\mu\nu}\, dp^\mu \wedge dq^\nu = \frac{i}{\omega_0} dp_\mu \wedge dq^\mu \tag{49}$$

The gauge potential λ upon application of D yields $F = \mathsf{D}\lambda$, the field strength 2-form. Moreover the right side of (49) is just

$$\left(\frac{1}{\omega_0}\Omega\right)(iI) = \frac{i}{\omega_0} dp_\mu \wedge dq^\mu \tag{50}$$

where $\Omega = dp_\mu \wedge dq^\mu$ is the symplectic form on phase space $\{(q, p)\}$ and (iI) is a $U(1)$ symmetry generator. This enables a restatement of the RCCR's in the form

"curvature of gauge bundle ≡ symplectic structure"

in much the same spirit as Caianiello's ideas.

The reciprocity principle fits naturally into the setup as follows. By demanding that the gauge potential λ be reciprocally invariant,

$$\rho^*(\lambda) = \lambda \tag{51}$$

one obtains reciprocally invariant decompositions of λ and F, namely,

$$\lambda = (\lambda^{\mathrm{GEOM}} + \lambda^{\mathrm{EM}})(iI) + \lambda^{\mathrm{INT}} \tag{52}$$

$$F = (F^{\mathrm{GEOM}} + F^{\mathrm{EM}})(iI) + F^{\mathrm{INT}} \tag{53}$$

Here λ^{EM}, the electromagnetic potential, is given by

$$\lambda^{\mathrm{EM}} = d\omega^{\mathrm{EM}} \tag{54}$$

with ω^{EM} a reciprocally invariant real-valued function leading to

$$F^{EM} = d\lambda^{EM} = 0 \tag{55}$$

and λ^{GEOM} is a reciprocally invariant 1-form whose exterior derivative yields the symplectic structure; for example,

$$\lambda^{GEOM} = \frac{1}{2\omega_0} g_{\mu\nu}(p^\mu dq^\nu - q^\nu dp^\mu) \tag{56}$$

$$F^{GEOM} = d\lambda^{GEOM} = \frac{1}{\omega_0}\Omega \tag{57}$$

Finally we also have the pure uncoupled Yang–Mills equation

$$F^{INT} = \mathrm{D}\lambda^{INT} = 0 \tag{58}$$

Thus, the general reciprocally invariant realization of the RCCR's is

$$\hat{Q}_\mu = -i\omega_0 \frac{\partial}{\partial p^\mu} + \frac{1}{2}\left(-q_\mu + \frac{\partial\omega}{\partial p^\mu}\right) - i\omega_0 \Phi_\mu^{INT} \tag{59a}$$

$$\hat{P}_\mu = i\omega_0 \frac{\partial}{\partial q^\mu} + \frac{1}{2}\left(-p_\mu - \frac{\partial\omega}{\partial q^\mu}\right) + i\omega_0 \Psi_\mu^{INT} \tag{59b}$$

where ω is a reciprocally invariant scalar function and Φ_μ^{INT}, Ψ_μ^{INT} define a gauge potential 1-form satisfying (58). Note also that reciprocal invariance of λ^{INT} provides the conditions

$$\Phi_\mu^{INT}(q, p) = \Psi_\mu^{INT}(p, -q), \qquad \Psi_\mu^{INT}(-q, -p) = -\Psi_\mu^{INT}(q, p) \tag{60}$$

sufficient to eliminate Φ_μ^{INT} from the description altogether, thereby allowing only as many gauge components as occur in standard quantum theory.

5. AN EXCITON MASS FORMULA

A very brief review of the arguments leading to a mass formula for excitons obeying Born's quantum metric operator eigenvalue equation is given. For details, see Brooke and Guz (1982, 1983b), Brooke and Prugovečki (1983).

Born's internal quantum metric operator is

$$\hat{D}^2 = \hat{Q}^\mu \hat{Q}_\mu + \hat{P}^\mu \hat{P}_\mu \tag{61}$$

with $\hat{Q}^\mu$, $\hat{P}^\mu$ satisfying the internal RCCR's (44) and $\omega_0 = \hbar/l_A m_A c$. The eigenvalue equation for the exciton transition amplitude $K_{B,A}$ is

$$\hat{D}^2 K_{B,A} = \lambda_{B,A} K_{B,A} \tag{62}$$

where the eigenvalues $\lambda_{B,A}$ are as in (30), (31).

Besides the external Klein–Gordon equation

$$\left[P^\mu P_\mu - \left(\frac{m_B}{m_A}\right)^2\right] K_{B,A} = 0 \tag{63}$$

which serves to define the masses that are observable, one requires another equation which couples the internal and external variables. Motivated by the Yukawa-type equation (27), we consider

$$\left[P^\mu P_\mu + \kappa\left(\hat{D}^2 - \omega_A\right) + \left(\frac{l_B}{l_A}\right)^2\right] K_{B,A} = 0 \tag{64}$$

where $P^\mu P_\mu$ represents the external part, κ is a coupling constant, and $(l_B/l_A)^2$ is a term suggested by reciprocal symmetry with $P^\mu P_\mu$ which, by (63), corresponds to $(m_B/m_A)^2$.

Combining (62), (63), (64), and using (31), one finds

$$\left(m_B^{\pm} c^2\right)^2 = \alpha^{-1}(2 + n_0 + 2n + J_B + \varepsilon) - \left(\frac{l_B}{l_A}\right)^2 \left(m_A c^2\right)^2 \tag{65}$$

where $\alpha = (2\hbar c\kappa)^{-1}(l_A/m_A c^2)$ represents the slope [J_B as a function of $(m_B c^2)^2$] of the trajectories implied by (65) for all values of $n_0, n = 0, 1, 2, \ldots$; and where ε takes the value 0 for natural parity states and the value 1 for unnatural parity states.

A consistency condition in the even parity case (namely, that $m_B = m_A$, $l_B = l_A$, $J_B = J_A$ when one sets $n_0 = n = 0$) results in

$$m_A c^2 = \sqrt{\alpha^{-1}\left(1 + \frac{J_A}{2}\right)} \tag{66}$$

Finally, comparing the qualitative aspects of formula (65) with experimental values of known baryon masses, one is prompted to set $l_B = l_A$ with the result that the final mass formula, obtainable from (65) (with $l_B = l_A$) and (66) is

$$\left(m_B^{\pm} c^2\right)^2 = \alpha^{-1}(n_0 + 2n + J_B - J_A + \varepsilon) + \left(m_A c^2\right)^2 \tag{67}$$

Experimentally it is known in the baryon case that $\alpha \approx 1(\mathrm{GeV})^{-2}$, and so if in equation (66) one sets $J_A = 1/2$ then the result is

$$m_{1/2}c^2 \approx 1154 \text{ MeV} \tag{68}$$

and if $J_A = 3/2$ then

$$m_{3/2}c^2 \approx 1365 \text{ MeV} \tag{69}$$

While there are no known spin-1/2 and spin-3/2 baryons of masses 1154 and 1365 MeV, respectively, it is true however that the *average* masses of the $SU(3)$ baryon octet: $p(938.3)-n(939.6)-\Lambda(1115.6)-\Sigma^+(1189.4)-\Sigma^0(1192.5)-\Sigma^-(1197.3)-\Xi^0(1314.9)-\Xi^-(1321.3)$ and decuplet: $\Delta^{++}(1226)-\Delta^+(1227)-\Delta^0(1231)-\Delta^-(1239)-\Sigma^{*+}(1382)-\Sigma^{*0}(1382)-\Sigma^{*-}(1387)-\Xi^{*0}(1532)-\Xi^{*-}(1535)-\Omega^-(1672)$ are very well predicted by (68) and (69), respectively. Taking this as evidence in support of the hypothesis that $SU(3)$ is (at least an approximate) internal symmetry group of the theory, one is led to regard the values of $m_{1/2}c^2$ and $m_{3/2}c^2$ as given by (66) as exact in an unbroken $SU(3)$ theory but only approximate in a broken $SU(3)$ theory.

Since the mass formula (67) is a result of the unbroken $SU(3)$ theory, one must decide how to interpret it in the broken theory. Because of the noncoupling (in this primitive model) with the electromagnetic field [equations (55), (58)], one should ignore in (67) contributions due to charged species and so one should take for the allowed values of $m_A c^2$ (the ground state energies of the exciton families) the values of $N(939)$, $\Sigma(1193)$, $\Lambda(1115)$, and $\Xi(1317)$ in the spin-1/2 case. And in the spin-3/2 situation, one should use for $m_A c^2$ the values of $\Delta(1232)$, $\Sigma^*(1385)$, $\Xi^*(1530)$, $\Omega(1672)$. Thus, using these as the ground state masses of the broken $SU(3)$ mass formula (67) one should obtain a mass spectrum for even and for odd parity trajectories. This set of predicted broken-$SU(3)$ exciton masses is in very good numerical agreement with the great majority of the known low-lying baryons.

A similar formula in the case of integral spin does not seem to fit experiment owing to the fact that the experimental meson trajectories are not parallel even within the tolerances allowed by experimental error.

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